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A Short Introduction to Constructive Algorithmics

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Constructive Algorithmics

- Constructive Algorithmics:
  "Methods for calculating programs from their specifications, and the design of notations for such calculation; also, the investigation of software support for the calculational process."
  - http://www.cs.auckland.ac.nz/research/groups/CDMTCS/docs/ca.html

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Outline

- Notations
  - BMF (Bird-Meertens Formalism)
  - Lists and their operations
- Program Calculation (Program Transformation)
  - Homomorphisms
  - Transformation rules
  - Calculation Example
- Extended Topics

Notations

Functions

- Function application is written without brackets
  - $f\ a$ means $f(a)$
- Function are curried and associate to the left
  - $f\ a\ b$ means $(f\ a)\ b$
- Function composition
  - $(f \cdot g)\ x = f(g\ x)$
- Binary operators can be sectioned
  - $(\oplus)\ a\ b = (a\oplus)\ b = (\oplus b)\ a = a \oplus b$

List

- Lists are finite sequence of values of the same type.
- Any list can be constructed with the following two operations:
  - Nil: An empty list
    - ✓ Denoted by [ ]
  - Cons: Put an element before a list
    - ✓ Denoted by a : as
- Example: [1, 2, 3] = 1 : (2 : (3 : [ ]))
- We may call such a list as "cons list"

Join List

- Another way to construct a list
- We use three operations to construct a list
  - Nil: An empty list
    - ✓ Denoted by [ ]
  - Singleton: A list with a single element
    - ✓ Denoted by [a] or [.] a
  - Join: Concatenation of two smaller lists
    - ✓ Denoted by $x \mathbin{+\!\!+} y$
    - ✓ Concatenation is associative

Map

- Map (denoted by *) takes a function and a list, and applies the function to each element of the list
- Informally:

$$f * [a_1, a_2, \ldots, a_n] = [f\ a_1, f\ a_2, \ldots, f\ a_n]$$

Reduce

- Reduce (denoted by /) takes an associative operator and a list, and collapse the list into a value with the operator.
- Informally:

$$\oplus/[a_1, a_2, \ldots, a_n] = a_1 \oplus a_2 \oplus \cdots \oplus a_n$$

Directed Reduce

- Left-to-right reduce (denoted by $\not\rightarrow$) is a reduce-like operation, but it can take non-associative operator.
- Informally:

$$\oplus \not\rightarrow_e [a_1, a_2, \ldots, a_n] \quad = \quad ((e \oplus a_1) \oplus \cdots \oplus a_n$$

- We can consider right-to-left reduce $\not\leftarrow$

Accumulation (Scan)

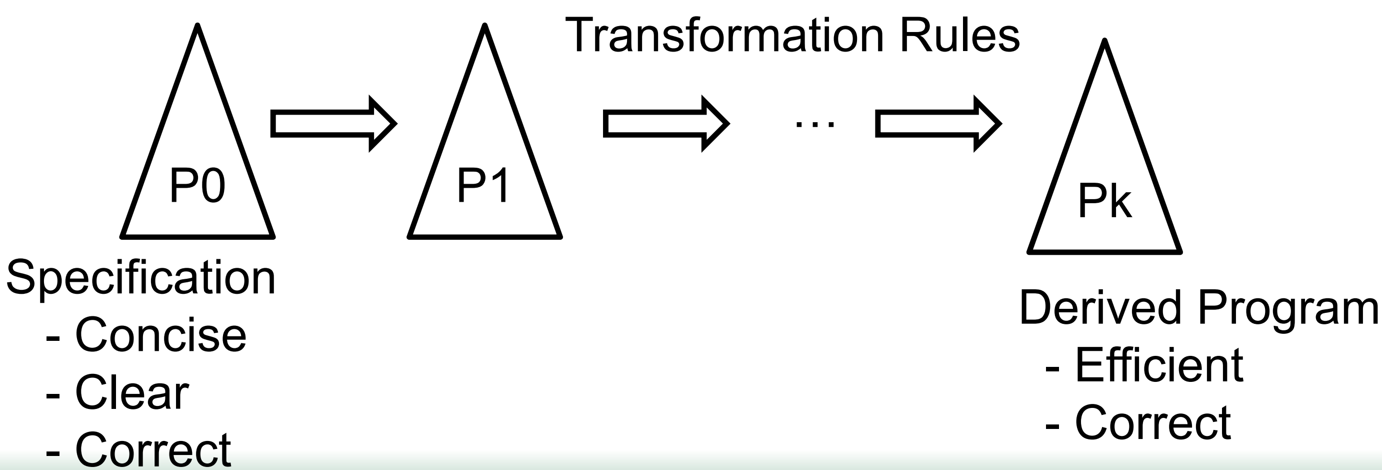
- Left-accumulate (denoted by $/\!\!\!/\!\!\!\!\rightarrow$) takes an operator and a list, and computes on the list by accumulating the values with the operator.
- Informally:

$$
\begin{aligned}
&\oplus /\!\!\!/\!\!\!\!\rightarrow_e [a_1, a_2, \ldots, a_n] \\
&\quad = \quad [e, e \oplus a, \ldots, ((e \oplus a_1) \oplus \cdots \oplus a_n]
\end{aligned}
$$

Program Calculation

Idea of Program Calculation

- Starting from a (mathematical) specification, we derive a program by applying transformation rules
  - Concise and clear specification
  - Transformation rules proved to be correct
  - Systematic / mechanical / automatic derivation



Homomorphism

- A function h defined in the following form is called homomorphism.

$$
\begin{aligned}
h\ [\,] &= id_{\oplus} \\
h\ [a] &= f\ a \\
h\ (x \mathbin{+\!\!+} y) &= h\ x \oplus h\ y
\end{aligned}
$$

- It maps a monoid ([α], ++, [ ]) to (β, $\oplus$, $id_{\oplus}$)
- Property: *h* is uniquely determined by f and $\oplus$

Homomorphism Theorems

- First homomorphism theorem:
  - Every homomorphism can be written as the composition of a map and a reduce.
  - $\mathrm{hom}\ (\oplus)\ f = (\oplus /)\ .\ (f\,*)$
- Third homomorphism theorem:
  - If a function h can be defined as
    $h\ ([a] \mathbin{++} x) = a \oplus h\ x$
    $h\ (y \mathbin{++} [b]) = h\ y \otimes b$
    then, h is a homomorphism.
  - This theorem also works as a parallelization theorem. [Morita et al., PLDI 2007]

Rules (1)

- Horner's rule:
  - E.g. $(a_1 \times a_2 \times a_3) + (a_2 \times a_3) + a_3 + 1 = ((1 \times a_1 + 1) \times a_2 + 1) \times a_3 + 1$
  - $\oplus / \cdot \otimes / * \cdot tails = \odot \not\to_e$
    - ✓ Distributivity over $\otimes$ and $\oplus$ is important
- Promotion:
  - $f * \cdot +\!\!+ / \quad = \quad +\!\!+ / \cdot (f*)*$
  - $\oplus / \cdot +\!\!+ / \quad = \quad \oplus / . (\oplus /)*$

Rules (2)

$$inits\ [a_1, a_2, \ldots, a_n] = [[\,], [a_1], [a_1, a_2], \ldots, [a_1, a_2, \ldots, a_n]]$$

- Accumulation:

$$(\oplus \mathbin{/\!\!\!/\!\!\!\rightarrow}_e) = (\oplus \not\rightarrow_e) * \cdot inits$$

Example (1)

- The maximum segment sum problem:
  Compute the maximum of sums of all segments of a given sequence of numbers

$$mss\ [3, 1, -4, 1, 5, -9, 2] = 6$$

- Concise and clear solution (specification)

$$mss = \uparrow / \cdot +/ * \cdot segs$$

  - But, slow: $O(n^3)$

Example (2)

- Calculating a linear-time algorithm

$$
\begin{array}{ll}
 & \uparrow / \cdot +/ * \cdot segs \\
= & \quad \{ \text{ definition of } segs \text{ } \} \\
 & \uparrow / \cdot +/ * \cdot +\!\!\!+ / \cdot tails * \cdot inits \\
= & \quad \{ \text{ map and reduce promotion } \} \\
 & \uparrow / \cdot (\uparrow / \cdot +/ * \cdot tails) * \cdot inits \\
= & \quad \{ \text{ Horner's rule with } a \odot b = (a+b) \uparrow 0 \text{ } \} \\
 & \uparrow / \cdot \odot \not\rightarrow_0 * \cdot inits \\
= & \quad \{ \text{ accumulation lemma } \} \\
 & \uparrow / \cdot \odot /\!\!\!/\!\!\!\!\rightarrow_0
\end{array}
$$

Extended Topics

List → Other data structures

- Matrix (2D array)
  - Data is constructed with Singleton, JoinX, JoinY
  - Some property is needed on JoinX/JoinY
- Binary tree
  - Data is constructed with Leaf and Node
    Node :: a -> (Tree a) -> (Tree a) → Tree a
- We can apply generic-programming technique (category-theoretic approach) to formalize other data structures (e.g., rose trees)

Relations

- Extend functions into relations
  - Studied by the group of Richard Bird (Oxford Univ.)
- Formalization of non-deterministic computations
- Example: Thinning theorem
  - Formalization of optimization problems (dynamic programming / approximation)
  - [Mu et al., WGP 2010.]

Optimization Problems

- Derive a dynamic-programming algorithm from the problem specification
  - Use algebraic properties of operators
- Examples
  - Maximum marking problem (Extension of mss)
    - ✓ Sasano et al. ICFP 2000
  - Generic maximum marking problem
    - ✓ Emoto et al. ESOP 2012

Parallel Programming

- Use of patterns as algorithmic skeletons
  - SkeTo parallel skeleton library
    - ✓ Data structures: lists, matrices, and trees
    - ✓ Skeletons: map, reduce, scan (and others)
- Third homomorphism theorem for parallelization
  - A tool for parallelization [Morita et al. PLDI 2007]
  - Parallelization for tree problems [Morihata et al. POPL 2009]

Tools

- Fusion optimization
  - Strong optimization in Haskell
  - Warm fusion: a powerful fusion rule based on list homomorphism
  - Stream fusion: a structure-generic fusion mechanism
- Yicho system
  - A tool for program calculation
  - Hylomorphism (extension of homomorphism)
  - [Yokoyama et al. APLAS 2002]

Conclusion

3 Important Aspects of Constructive Algorithmics

- Notations and patterns
  - Bird-Meertens Formalism (BMF)
  - Functional modeling of
- Program Calculations
  - Systematically deriving programs from the specification
  - Theorems for homomorphisms
- Supporting Tools
  - For systematic/automatic derivation of programs
  - Domain-specific tools for program derivation

References

- Lecture notes about constructive algorithmics by Zhenjiang Hu (NII Japan)
  - http://research.nii.ac.jp/~hu/pub/teach/pm06/CA[1-4].pdf